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The limit states of magnetic relaxation

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It is generally believed that a viscous, non-resistive plasma will eventually decay to a magnetostatic state, probably possessing contact discontinuities. We prove that even in the presence of a decaying forcing, the kinetic energy of the system tends to zero, which justifies the belief that the limit state is static. Regarding the magnetic field, the fact that the magnetic energy remains bounded proves the existence of weak sequential limits of the field as the time goes to infinity, but this does not imply that the magnetic field tends to a single state: we present an example where there is no limit, even in a weak sense. One additional condition upon the velocity, however, is enough to guarantee existence of a single limit magnetic configuration.

1. Introduction

In a classical paper Moffatt (1985) described the evolution of an infinitely conducting, viscous, incompressible fluid according to the laws of magnetohydrodynamics (MHD):

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = \nu\Delta\boldsymbol{v} + \boldsymbol{J}\times\boldsymbol{B} - \nabla p, \tag{1}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{v}\times\boldsymbol{B}), \tag{2}$$

$$\nabla\cdot\boldsymbol{v} = \nabla\cdot\boldsymbol{B} = 0. \tag{3}$$

Here $\boldsymbol{v}$ represents the fluid velocity, $\boldsymbol{B}$ the magnetic field, $\boldsymbol{J}=\nabla\times\boldsymbol{B}$ the current density, p the kinetic pressure and ν the viscosity. We have normalized the density to unity for simplicity of notation.

The boundary conditions imposed in Moffatt (1985) are Dirichlet ones: both $\boldsymbol{v}$ and the normal component of $\boldsymbol{B}$ vanish at the boundary of the smooth domain Ω for all time. Moffatt then proves the energy identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_\Omega v^2 + B^2\,\mathrm{d}V\right) = -\nu\int_\Omega |\nabla\boldsymbol{v}|^2\,\mathrm{d}V, \tag{4}$$

which means that the total energy decreases monotonically for as long as $\boldsymbol{v}$ is not zero. The conclusion is that the limit state of the velocity when $t\rightarrow\infty$ is zero, and therefore the field must tend to a magnetostatic state where $\boldsymbol{J}\times\boldsymbol{B}=\nabla p$. The topology of the magnetic field lines is conserved for all time, because the ideal induction equation (2) implies that field lines are transported by the flow as material points, but in the limit the topology may change and indeed the field may become discontinuous. This is a welcome feature, since contact discontinuities in the form of current sheets are extremely useful in explaining ubiquitous phenomena involving rapid conversion of magnetic to kinetic energy: Parker (1994) made this one of the cornerstones of his theory. On the mathematical side, variants of Moffatt's setting have been proposed to prove existence of equilibria for the Euler equations (Nishiyama 2002, 2003).

Moffatt's paper abounds with well-chosen examples of how the field line configuration may obstruct the decay to zero of the total energy, and it is a model of succesful combination of intuitive description with rigorous argument. It does not, however, prove that the velocity and the magnetic field really tend to magnetostatic equilibria, or indeed that they tend to anything. The simple decrease of total energy is not strong enough to guarantee existence of asymptotic states. We will analyse this subject, and prove that even in the presence of a forcing and with more general boundary conditions, the kinetic energy does in fact tend to zero, which may be interpreted as saying that the limit state is stationary. The magnetic field, however, is more difficult: it becomes arbitrarily close to a certain set of states, at least in a weak sense; but these limits do not need to be unique, except with an additional condition upon the velocity. That this condition, although reasonable, does not hold in all cases is proved with an example where there is no limit of the magnetic field.

In the absence of any forcing, even if decaying, it is more difficult to show any example of such behaviour, mainly because of the lack of non-trivial analytic solutions to the MHD equations. Only the simplest instances, such as plane flow and vertical field, or velocity and field aligned, provide simple solutions and for these the velocity decays exponentially and therefore satisfies every possible integrability condition. Thus, we have not strictly refuted the belief that the magnetic field relaxes to a specific state in an unforced flow. However, since our forcings tend to zero for large times, we have shown that the smallest of errors in the integration of the momentum equation may destroy the possible convergence of the magnetic field; hence this convergence is not robust. In fact, we conjecture that there is no such convergence.

We explain briefly our (standard) notation: $L^p(\Omega)$ will denote the space of functions (or vector functions) such that $|f|^p$ is integrable, with the norm

$$\|f\|_p = \left(\int_\Omega |f|^p \, \mathrm{d}V \right)^{1/p},$$

whereas $L^\infty(\Omega)$ is the space of measurable functions, bounded except perhaps in a set of null measure, with the maximum norm. $L^p((0,\infty), L^q(\Omega))$ will logically denote the space of measurable functions g of time taking values in $L^q(\Omega)$, such that

$$\|g\|_{L^p((0,\infty),L^q(\Omega))} = \left(\int_0^\infty \|g(t)\|_q^p \, \mathrm{d}t \right)^{1/p} < \infty.$$

Of course a function $g(t, \boldsymbol{x})$ of time and space variables may always be interpreted as a function of time, taking values in a space of functions in $\boldsymbol{x}$. This second interpretation is often more convenient for simplicity of notation. $\mathscr{C}_c^\infty(\Omega)$ is the space of test functions, i.e. those infinitely differentiable of compact support contained in Ω.

For all the spaces we will be considering, corresponding to the different boundary conditions upon the velocity, a *Poincaré inequality* will hold, i.e. there will exist a constant C depending only on the domain Ω such that

$$\|\boldsymbol{v}\|_2 \leqslant C \|\nabla \boldsymbol{v}\|_2.$$

This occurs whenever Ω is smooth and bounded, and there exists a seminorm p such that p never vanishes on non-zero constant functions, whereas for all the functions within our space, $p(f)=0$. Examples of such seminorms are

$$p(\boldsymbol{v}) = \sup_{\partial\Omega} |\boldsymbol{v}\cdot\boldsymbol{n}|, \tag{5}$$

which includes Dirichlet conditions, and

$$p(\boldsymbol{v}) = \left| \int_{\Omega} \boldsymbol{v}\, \mathrm{d}V \right|, \tag{6}$$

used for periodic problems. This result is proved in Deny & Lions (1954); see also Temam (1980, pp. 49–50).

We will assume that there exists a smooth solution $(\boldsymbol{v}, \boldsymbol{B})$ defined for all time, satisfying in addition that the magnetic field remains bounded in the L^{∞}-norm. Neither of these hypotheses is a theorem: there is no proof of the global existence of smooth solutions to the diffusive MHD equations, much less if we omit the resistivity. Still, it does not make sense otherwise to study the limit $t\to\infty$. The assumption $\|\boldsymbol{B}\|_{\infty} \leqslant M$ also seems physically reasonable.

2. Asymptotic behaviour of the velocity

The forced non-resistive MHD system may be written as

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = \nu\Delta\boldsymbol{v} + \boldsymbol{B}\cdot\nabla\boldsymbol{B} - \nabla p_* + \boldsymbol{f}, \tag{7}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{B} = \boldsymbol{B}\cdot\nabla\boldsymbol{v}, \tag{8}$$

$$\nabla\cdot\boldsymbol{v} = \nabla\cdot\boldsymbol{B} = 0, \tag{9}$$

where $p_* = p + (1/2)B^2$ is the total pressure. We will not demand any specific boundary conditions, except that the following boundary integrals must vanish:

$$\int_{\partial\Omega} \boldsymbol{v}\cdot\frac{\partial \boldsymbol{v}}{\partial n}\,\mathrm{d}\sigma = \int_{\partial\Omega} (\boldsymbol{B}\cdot\boldsymbol{v})(\boldsymbol{B}\cdot\boldsymbol{n})\,\mathrm{d}\sigma$$
$$= \int_{\partial\Omega} v^2(\boldsymbol{v}\cdot\boldsymbol{n})\,\mathrm{d}\sigma = \int_{\partial\Omega} B^2(\boldsymbol{v}\cdot\boldsymbol{n})\,\mathrm{d}\sigma = \int_{\partial\Omega} p(\boldsymbol{v}\cdot\boldsymbol{n})\,\mathrm{d}\sigma = 0. \tag{10}$$

This holds for example: with Dirichlet ($\boldsymbol{v}=\boldsymbol{0}$ at $\partial\Omega$) conditions for the velocity, for any value of $\boldsymbol{B}$; when $\boldsymbol{v}\cdot\boldsymbol{n}=\boldsymbol{B}\cdot\boldsymbol{v}=(\partial v^2/\partial n)=0$ at $\partial\Omega$; for periodic boundary conditions; or for several combinations of those. $\boldsymbol{n}$ denotes the normal vector at the boundary.

In addition we assume that a Poincaré inequality holds. This occurs e.g. when $\boldsymbol{v}\cdot\boldsymbol{n}=0$ or when the integral of $\boldsymbol{v}$ in Ω is zero.

Notice that we do not impose any condition on the magnetic field. This is because it is not really necessary and the traditional one, $\boldsymbol{B}\cdot\boldsymbol{n}=0$, occurring in Moffatt (1985) is not adequate for relaxation problems, where the anchoring of field lines at the boundary is an important characteristic (Low & Wolfson 1988). With Dirichlet conditions upon the velocity the value of $\boldsymbol{B}\cdot\boldsymbol{n}$ at $\partial\Omega$ remains constant in time. Concerning the forcing $\boldsymbol{f}$, we will assume that $\boldsymbol{f}\in L^2((0,\infty), L^2(\Omega))$, i.e. that the energy of $\boldsymbol{f}$ is integrable in time.

Let us analyse the energy inequality. By multiplying the momentum equation (7) by $\boldsymbol{v}$, the induction equation (8) by $\boldsymbol{B}$, adding and applying the divergence theorem, we find as usual

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\|\boldsymbol{v}\|_2^2 + \|\boldsymbol{B}\|_2^2\right) = -\nu\|\nabla\boldsymbol{v}\|_2^2 + (\boldsymbol{f}, \boldsymbol{v}), \tag{11}$$

where

$$(\boldsymbol{f}, \boldsymbol{v}) = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}V.$$

By the inequalities of Cauchy-Schwarz and Young,

$$|(\boldsymbol{f}, \boldsymbol{v})| \leqslant \|\boldsymbol{f}\|_2 \|\boldsymbol{v}\|_2 \leqslant C \|\boldsymbol{f}\|_2 \|\nabla \boldsymbol{v}\|_2 \leqslant \frac{\nu}{2} \|\nabla \boldsymbol{v}\|_2^2 + \frac{C^2}{2\nu} \|\boldsymbol{f}\|_2^2,$$

where C denotes the Poincaré constant. Hence

$$\frac{1}{2} \frac{\partial}{\partial t} \left(\|\boldsymbol{v}\|_2^2 + \|\boldsymbol{B}\|_2^2 \right) \leqslant \frac{-\nu}{2} \|\nabla \boldsymbol{v}\|_2^2 + \frac{C^2}{2\nu} \|\boldsymbol{f}\|_2^2.$$

Integrating in time,

$$\frac{1}{2} \left(\|\boldsymbol{v}(t)\|_2^2 + \|\boldsymbol{B}(t)\|_2^2 \right) \leqslant \frac{1}{2} \left(\|\boldsymbol{v}(0)\|_2^2 + \|\boldsymbol{B}(0)\|_2^2 \right) - \frac{\nu}{2} \int_0^t \|\nabla \boldsymbol{v}\|_2^2 \, \mathrm{d}s + \frac{C^2}{2\nu} \int_0^t \|\boldsymbol{f}\|_2^2 \, \mathrm{d}s. \tag{12}$$

Since we assumed that $\boldsymbol{f} \in L^2((0, \infty), L^2(\Omega))$, the last integral is uniformly bounded for all t. Therefore $\|\boldsymbol{v}\|_2$, $\|\boldsymbol{B}\|_2$ are bounded for all time, and the function $\|\nabla \boldsymbol{v}\|_2^2$ is also integrable in $(0, \infty)$. This implies that $\|\nabla \boldsymbol{v}\|_2$ (and *a fortiori* $\|\boldsymbol{v}\|_2$) is mostly small for advanced times, but in principle it may have jumps of arbitrarily large height provided their durations become rapidly smaller. Indeed, one could suspect that this may occur, say, if the magnetic energy is converted into kinetic energy in short bursts. The total energy would decrease, but each of its summands could be unpredictable. We will prove that this does not occur, and that $\lim_{t \to \infty} \|\boldsymbol{v}(t)\|_2 = 0$. Thus the system tends to become stationary.

The proof follows the lines of an analogous one for the case where Ω is the whole space, and in the absence of forcing, in Agapito & Schonbek (2006). Our proof is simpler because of the help of the Poincaré inequality.

Take a fixed $r > 0$ such that $rC^2 \leqslant \nu$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{e}^{rt} \|\boldsymbol{v}(t)\|_2^2 \right) = r\mathrm{e}^{rt} \|\boldsymbol{v}(t)\|_2^2 - \nu \mathrm{e}^{rt} \|\nabla \boldsymbol{v}(t)\|_2^2 + \mathrm{e}^{rt} (\boldsymbol{B} \cdot \nabla \boldsymbol{B}, \boldsymbol{v}) + \mathrm{e}^{rt} (\boldsymbol{f}, \boldsymbol{v}), \tag{13}$$

since

$$(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{v}) = \frac{1}{2} \int_{\Omega} \boldsymbol{v} \cdot \nabla v^2 \, \mathrm{d}V = \frac{1}{2} \int_{\partial \Omega} v^2 \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}\sigma = 0,$$

and

$$(\nabla p_*, \boldsymbol{v}) = \int_{\partial \Omega} p_* \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}\sigma = 0.$$

By an analogous argument, $(\boldsymbol{B} \cdot \nabla \boldsymbol{B}, \boldsymbol{v}) = -(\boldsymbol{B} \cdot \nabla \boldsymbol{v}, \boldsymbol{B})$. Therefore, using Poincaré's as well as other standard inequalities,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{e}^{rt} \|\boldsymbol{v}(t)\|_2^2 \right) &\leqslant rC^2 \mathrm{e}^{rt} \|\nabla \boldsymbol{v}(t)\|_2^2 - \nu \|\nabla \boldsymbol{v}(t)\|_2^2 \\ &\quad + \mathrm{e}^{rt} \|\boldsymbol{B}(t)\|_\infty \|\boldsymbol{B}(t)\|_2 \|\nabla \boldsymbol{v}(t)\|_2 + C\mathrm{e}^{rt} \|\boldsymbol{f}(t)\|_2 \|\nabla \boldsymbol{v}(t)\|_2 \\ &\leqslant \mathrm{e}^{rt} \|\nabla \boldsymbol{v}(t)\|_2 (\|\boldsymbol{B}\|_\infty \|\boldsymbol{B}\|_2 + C \|\boldsymbol{f}(t)\|_2). \end{aligned} \tag{14}$$

Since by hypothesis $\|\boldsymbol{B}\|_\infty$ and (*a fortiori*, or by the energy inequality) $\|\boldsymbol{B}\|_2$ are bounded, we find that for some constant k

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{rt}\|\boldsymbol{v}(t)\|_2^2\right) \leqslant \mathrm{e}^{rt}\|\nabla\boldsymbol{v}(t)\|_2(k + C\|\boldsymbol{f}(t)\|_2). \tag{15}$$

Integrating between s and t, and multiplying by e^{-rt}, we find

$$\begin{aligned}\|\boldsymbol{v}(t)\|_2^2 \leqslant\ & \mathrm{e}^{r(s-t)}\|\boldsymbol{v}(s)\|_2^2 + k\mathrm{e}^{-rt}\int_s^t \mathrm{e}^{rx}\|\nabla\boldsymbol{v}(x)\|_2\,\mathrm{d}x \\ & + C\int_s^t \mathrm{e}^{r(x-t)}\|\boldsymbol{f}(x)\|_2\|\nabla\boldsymbol{v}(x)\|_2\,\mathrm{d}x \\ \leqslant\ & \mathrm{e}^{r(s-t)}\|\boldsymbol{v}(s)\|_2^2 + k\mathrm{e}^{-rt}\left(\int_s^t \mathrm{e}^{2rx}\,\mathrm{d}x\right)^{1/2}\left(\int_s^t \|\nabla\boldsymbol{v}(x)\|_2^2\,\mathrm{d}x\right)^{1/2} \\ & + C\left(\int_s^t \|\boldsymbol{f}(x)\|_2^2\,\mathrm{d}x\right)^{1/2}\left(\int_s^t \|\nabla\boldsymbol{v}(x)\|_2^2\,\mathrm{d}x\right)^{1/2}.\end{aligned} \tag{16}$$

Therefore, bounding $\|\boldsymbol{v}(s)\|_2$ by a bound of the total energy A:

$$\begin{aligned}\|\boldsymbol{v}(t)\|_2^2 \leqslant\ & \mathrm{e}^{r(s-t)}A + \frac{k}{\sqrt{2r}}\left(\int_s^t \|\nabla\boldsymbol{v}(x)\|_2^2\,\mathrm{d}x\right)^{1/2} \\ & + C\left(\int_0^\infty \|\boldsymbol{f}(x)\|_2^2\,\mathrm{d}x\right)^{1/2}\left(\int_s^t \|\nabla\boldsymbol{v}(x)\|_2^2\,\mathrm{d}x\right)^{1/2}.\end{aligned} \tag{17}$$

For any arbitrary constant $\epsilon > 0$, if we take s_0 large enough for

$$\mathrm{e}^{-rs_0} < \epsilon, \qquad \int_{s_0}^\infty \|\nabla\boldsymbol{v}(x)\|_2^2\,\mathrm{d}x < \epsilon^2,$$

we find that for any $t > 2s_0$,

$$\|\boldsymbol{v}(t)\|_2^2 \leqslant \left(A + \frac{k}{\sqrt{2r}} + Cm\right)\epsilon, \tag{18}$$

where m is the integral of $\|\boldsymbol{f}\|_2^2$ in $(0,\infty)$. This proves that $\|\boldsymbol{v}\|_2$ becomes arbitrarily small for large times. The result is proved.

(Note: the Poincaré inequality may be refined by using instead the Gagliardo–Nirenberg inequality. For our purposes, this would imply that it is enough to have

$$\boldsymbol{f} \in L^2((0,\infty), L^{6/5}(\Omega)),$$

for all our results to hold (in dimension three). Still, this does not seem a major improvement.)

3. Analysis of the magnetic field evolution

The convergence of the velocity to zero implies that $\|\boldsymbol{B}\|_2$ has a limit when $t\to\infty$; this is

$$\lim_{t\to\infty}\|\boldsymbol{B}(t)\|_2^2 = \|\boldsymbol{B}(0)\|_2^2 + \|\boldsymbol{v}(0)\|_2^2 - 2\nu\int_0^\infty \|\nabla\boldsymbol{v}(s)\|_2^2\,\mathrm{d}s + 2\int_0^\infty (\boldsymbol{f},\boldsymbol{v})\,\mathrm{d}s. \tag{19}$$

The boundedness of the set $\{\boldsymbol{B}(t) : t \geqslant 0\}$ in $L^2(\Omega)$ implies that this set of functions is weakly relatively compact and metrizable, by the theorem of Alaoglu. We do not need to bother with the deeper meaning of this result: what concerns us is that every sequence s_n tending to ∞ contains a subsequence t_n such that $\boldsymbol{B}(t_n)$ converges weakly to some $\boldsymbol{B}_0 \in L^2(\Omega)$, i.e. for every $\boldsymbol{w} \in L^2(\Omega)$,

$$(\boldsymbol{B}(t_n), \boldsymbol{w}) \to (\boldsymbol{B}_0, \boldsymbol{w}). \tag{20}$$

The classical example to show the difference between weak and L^2 convergence involves the trigonometric functions $e^{i\boldsymbol{m}\cdot\boldsymbol{x}}$, which tend weakly to zero when $|\boldsymbol{m}| \to \infty$, although they have a constant norm. In our case, if the magnetic fields have Fourier modes only in an interval tending to ∞ with time, they tend weakly to zero.

If this limit were unique this would mean that the whole function $\boldsymbol{B}(t)$ tends to $\boldsymbol{B}_0$ weakly, which is at least partially satisfactory for the existence of a limit state. However, this is not guaranteed in principle, although it is true that if another sequence t'_n does not separate too much from t_n, then $\boldsymbol{B}(t'_n)$ has the same weak limit; and that if the velocity satisfies a certain bound, then the limit is unique. Both results follow from the weak form of the induction equation: for any test function $\boldsymbol{w} \in \mathscr{C}_c^\infty(\Omega)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{B}(t), \boldsymbol{w}) = (\boldsymbol{v} \times \boldsymbol{B}, \nabla \times \boldsymbol{w}), \tag{21}$$

which follows easily by multiplying the original equation by $\boldsymbol{w}$ and applying integral theorems. Integrating in time and applying simple inequalities:

$$|(\boldsymbol{B}(t) - \boldsymbol{B}(s), \boldsymbol{w})| \leqslant \|\nabla \times \boldsymbol{w}\|_\infty \int_s^t \|\boldsymbol{v}(x)\|_2 \|\boldsymbol{B}(x)\|_2 \,\mathrm{d}x. \tag{22}$$

If we know *a priori* that $|t'_n - t_n| \leqslant T$ for a fixed constant T, then

$$\begin{aligned} |(\boldsymbol{B}(t'_n) - \boldsymbol{B}(t_n), \boldsymbol{w})| &\leqslant \|\nabla \times \boldsymbol{w}\|_\infty \left(\sup_x \|\boldsymbol{B}(x)\|_2\right) \int_{t_n}^{t'_n} \|\boldsymbol{v}(x)\|_2 \,\mathrm{d}x \\ &\leqslant \|\nabla \times \boldsymbol{w}\|_\infty \left(\sup_x \|\boldsymbol{B}(x)\|_2\right) \sqrt{T} \left(\int_{t_n}^{t'_n} \|\boldsymbol{v}(x)\|_2^2 \,\mathrm{d}x\right)^{1/2}, \end{aligned} \tag{23}$$

and since the function $\|\boldsymbol{v}\|_2^2$ is integrable in $(0, \infty)$ by Poincaré's inequality, the last integral is arbitrarily small for t_n advanced enough. Since the set of test functions is dense in $L^2(\Omega)$, any weak limit of $\boldsymbol{B}(t'_n)$ (or of any subsequence) must again be the limit $\boldsymbol{B}_0$ of $\boldsymbol{B}(t_n)$. Therefore both sequences have the same weak limit.

Notice that the existence of the constant T is fundamental; we cannot guarantee that sequences at widely different times have the same limit. We can do that, and prove the existence of a unique weak limit, if we assume $\boldsymbol{v} \in L^1(0, \infty), L^1(\Omega))$:

$$|(\boldsymbol{B}(t) - \boldsymbol{B}(s), \boldsymbol{w})| \leqslant \|\nabla \times \boldsymbol{w}\|_\infty (\sup_x \|\boldsymbol{B}(x)\|_\infty) \int_s^t \|\boldsymbol{v}(x)\|_1 \,\mathrm{d}x, \tag{24}$$

and the last integral in (23) becomes arbitrarily small. Hence the limit is unique, which implies that $\boldsymbol{B}(t) \to \boldsymbol{B}_0$ weakly when $t \to \infty$.

Is the condition $\boldsymbol{v} \in L^1(0, \infty), L^1(\Omega))$ also necessary for the existence of the limit? There are some partial intuitive arguments in this sense: first let us prove that if this condition does not hold, there are fluid trajectories of arbitrarily large length. Recall that the particle path $\boldsymbol{\xi}(\boldsymbol{a}, t)$ starting at the point $\boldsymbol{a}$ satisfies

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t}(\boldsymbol{a}, t) = \boldsymbol{v}(\boldsymbol{\xi}(\boldsymbol{a}, t), t), \quad \boldsymbol{\xi}(\boldsymbol{a}, 0) = \boldsymbol{a}. \tag{25}$$

Since $\nabla \cdot \boldsymbol{v} = 0$, the mapping $\boldsymbol{a} \to \boldsymbol{\xi}(\boldsymbol{a}, t)$ for fixed t is a diffeomorphism of Ω which leaves the volume invariant, i.e. its Jacobian determinant is one. Thus

$$\int_\Omega |\boldsymbol{v}(\boldsymbol{x}, t)| \, \mathrm{d}V(\boldsymbol{x}) = \int_\Omega |\boldsymbol{v}(\boldsymbol{\xi}(\boldsymbol{a}, t), t)| \, \mathrm{d}V(\boldsymbol{a}) = \int_\Omega \left| \frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t}(\boldsymbol{a}, t) \right| \mathrm{d}V(\boldsymbol{a}). \tag{26}$$

Hence

$$\begin{aligned} \|\boldsymbol{v}\|_{L^1((0,\infty),L^1(\Omega))} &= \int_0^\infty \mathrm{d}t \int_\Omega |\boldsymbol{v}(\boldsymbol{x}, t)| \, \mathrm{d}V(\boldsymbol{x}) \\ &= \int_\Omega \mathrm{d}V(\boldsymbol{a}) \int_0^\infty \left| \frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t}(\boldsymbol{\xi}(\boldsymbol{a}, t), t) \right| \mathrm{d}t = \int_\Omega \Lambda(\boldsymbol{a}) \, \mathrm{d}V(\boldsymbol{a}), \end{aligned} \tag{27}$$

where $\Lambda(\boldsymbol{a})$ denotes the length of the trajectory starting at the point $\boldsymbol{a}$, $t \to \boldsymbol{\xi}(\boldsymbol{a}, t)$. Notice that while $\boldsymbol{a} \to \Lambda(\boldsymbol{a})$ is a measurable function by Fubini's theorem, it is not clear if it is continuous, since particle paths starting at nearby points may differ widely for large times. Anyway, it is obvious that this function cannot be bounded in Ω if $\|\boldsymbol{v}\|_{L^1((0,\infty),L^1(\Omega))} = \infty$.

Since the magnetic field is transported by the flow, one could think of tagging a point of each trajectory with a characteristic magnetic field vector; in the case that this norm of the velocity is infinite, one never reaches a limit of all particle paths asymptotically in time, so the field never comes or tends to a stop. It therefore cannot have a limit when $t \to \infty$.

Of course the above picture has several problems. One is that it seems difficult to set a magnetic vector for every point of Ω that is recognizable and distinct from the others as it is transported by the flow. The most important one, however, is that the field itself affects the flow through the Lorentz force in the momentum equation, so we cannot consider it a passive scalar.

In some instances, however, this is precisely what happens. Consider a domain of the form $\Omega = U \times (0, R)$, a plane velocity field of the form $\boldsymbol{v} = (v_1(t, x, y), v_2(t, x, y), 0)$ satisfying Dirichlet or periodic conditions in U, and a vertical magnetic field $\boldsymbol{B} = (0, 0, b(x, y, t))$. Notice that all boundary conditions (10) are satisfied, since $\boldsymbol{B} \cdot \boldsymbol{v} = 0$ everywhere, and $\partial \boldsymbol{v}/\partial n = \boldsymbol{0}$ in the horizontal boundaries $U \times \{0, R\}$. This configuration remains valid for all time, since the Lorentz force is also plane:

$$\boldsymbol{J} \times \boldsymbol{B} = -\tfrac{1}{2} \nabla b^2. \tag{28}$$

The magnetic field, as expected, is transported by the flow as a passive scalar: the induction equation is merely

$$\frac{\partial b}{\partial t} + \boldsymbol{v} \cdot \nabla b = 0, \tag{29}$$

and it does not affect the flow, since we can always merge the Lorentz force with the pressure term. The momentum equation is

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = \nu \Delta \boldsymbol{v} - \nabla p_* + \boldsymbol{f}. \tag{30}$$

This is the Navier–Stokes equation, except for the fact that we now have the total pressure. It may be solved by projecting into the space of functions of divergence zero appropriate to the boundary conditions ($\boldsymbol{v} \cdot \boldsymbol{n} = 0$ in the Dirichlet case; $\boldsymbol{v} \cdot \boldsymbol{n}$ antiperiodic, $\boldsymbol{v}$ of mean zero in the periodic case; see e.g. Temam 1980). This projection kills the pressure gradient. The pressure may be recovered afterwards once the velocity is known. Thus, no matter what the value of b, we may simply set the kinetic pressure p so that $p + b^2/2$ is the value p_* found after solving the equation.

This construction satisfies the above requirements. If we can find an example such that $\boldsymbol{v} \notin L^1((0,\infty), L^1(\Omega))$, we could find a magnetic field without limit as $t \to \infty$. Unfortunately, in the absence of forcing the Navier–Stokes equation in a bounded domain yields exponential decrease of the kinetic energy to zero, and an exponentially decreasing function is as integrable as one can wish. (Things are very different in the whole space: here one obtains at most algebraic decrease, see e.g. Kukavica 2001). Still, the bounded case is more relevant to our study because of the important role played by the anchoring of magnetic field lines at the boundary in magnetic relaxation). Notice, however, that we have allowed a forcing $\boldsymbol{f} \in L^2((0,\infty), L^2(\Omega))$ in our results, and this will be enough to obtain an example.

Let $U = (0, 2\pi) \times (0, 2\pi)$. We will take functions of the form

$$\boldsymbol{v} = (v(y,t), 0, 0), \qquad \boldsymbol{B} = (0, 0, b(x,y,t)), \qquad \boldsymbol{f} = (f(y,t), 0, 0).$$

All of them must be smooth, periodic, and of zero mean in U. In this case the pressure gradient vanishes and the MHD equations (29) and (30) become

$$\frac{\partial v}{\partial t} - \nu \frac{\partial^2 v}{\partial y^2} = f, \tag{31}$$

$$\frac{\partial b}{\partial t} + v \frac{\partial b}{\partial x} = 0. \tag{32}$$

Let us start with the induction equation (32). Let $W(y,t) = \int_0^t v(y,s)\,\mathrm{d}s$ denote a primitive of v. For the initial condition

$$b(x, y, 0) = \cos x, \tag{33}$$

the solution of (32) is

$$b(x, y, t) = \cos(x - W(y,t)). \tag{34}$$

Let us take a velocity of the form

$$v(y,t) = \lambda(t)\phi(y), \tag{35}$$

where ϕ is a smooth periodic function of mean zero such that it is constant (e.g. $\phi = 1$) in a certain subinterval $I \subset (0, 2\pi)$. λ will be a positive function such that both it and its differential λ' belong to $L^2(0,\infty)$, but $\lambda \notin L^1(0,\infty)$; an example is $\lambda(t) = 1/(1+t)$.

Take now

$$f(y,t) = \lambda'(t)\phi(y) - \nu\lambda(t)\phi_{yy}(y). \tag{36}$$

Thus $f \in L^2((0,\infty), L^2(\Omega))$, but $f \notin L^1((0,\infty), L^1(\Omega))$. The solution of the momentum equation (31) with initial condition

$$v(y, 0) = \lambda(0)\phi(y) \tag{37}$$

is precisely $v(y,t) = \lambda(t)\phi(y)$.

In this case $W(y,t) = \Lambda(t)\phi(y)$, where $\Lambda(t) = \int_0^t \lambda(s)\,\mathrm{d}s$ (in our example, $\Lambda(t) = \log(t+1)$). Then $\Lambda(t) \to \infty$ when $t \to \infty$, although the convergence is slow. Let us show that the magnetic field

$$b(x,y,t) = \cos(x - W(y,t)) = \cos(x - \Lambda(t)\phi(y)) \tag{38}$$

illustrates all the concepts discussed above. First, the integral

$$\iint_U b(x, y, t)^2 \, \mathrm{d}x \, \mathrm{d}y = \iint_U \cos^2(x - \Lambda(t)\phi(y)) \, \mathrm{d}x \, \mathrm{d}y \tag{39}$$

does not depend on t. This may be shown by a change of variables

$$r = x - \Lambda(t)\phi(y), \quad y = y,$$

and using the fact that the integral of a 2π-periodic function in an interval of length 2π is always the same:

$$\int_{-\Lambda(t)\phi(y)}^{2\pi - \Lambda(t)\phi(y)} \cos^2 r \, \mathrm{d}r = \int_0^{2\pi} \cos^2 r \, \mathrm{d}r = \pi.$$

Hence the magnetic energy $\|\boldsymbol{B}\|_2 = \|b\|_2 = \pi\sqrt{2}$ is constant in time, so that obviously it has a limit when $t \to \infty$. Let us show that the field itself has no limit, even in a weak sense. It is clearly enough to show that neither $\cos W(y, t)$ nor $\sin W(y, t)$ have weak limits in L^2 when $t \to \infty$. Take a test function ψ localized in the interval I: there $W(y, t) = \Lambda(t)$. Thus

$$\int_0^{2\pi} (\cos W(y, t))\psi(y) \, \mathrm{d}y = (\cos \Lambda(t)) \int_I \psi(y) \, \mathrm{d}y. \tag{40}$$

Since $\Lambda(t) \to \infty$ when $t \to \infty$, $\cos \Lambda(t)$ (resp. $\sin \Lambda(t)$) oscillates between the values -1 and 1, never tending to either of them. Since the mean of ψ does not need to be zero, the previous integral has no limit.

Notice also that for a sequence t_n such that $\Lambda(t_n) = \theta + 2h_n\pi$ for a fixed θ and integer h_n, the integral is in fact constant. This illustrates (at least for this type of test function) the result about weak convergence of subsequences.

Hypotheses upon the forcing (or more accurately, upon the solenoidal part of the forcing) guaranteeing that the velocity is integrable in time exist (see Lions 1996, pp. 92–110), but they are highly technical, involving Hardy and Lorentz spaces, and they would contribute little to our understanding of the problem. Certainly, if we could prove that the solenoidal projection of $\boldsymbol{J} \times \boldsymbol{B}$ belongs to one of these spaces it would imply the weak convergence of the magnetic field as $t \to \infty$, but this looks harder than the original problem.

Notice that even if this limit exists, we should prove $\partial \boldsymbol{v}/\partial t \to \boldsymbol{0}$, $\boldsymbol{v} \cdot \nabla \boldsymbol{v} \to \boldsymbol{0}$ in some sense to guarantee that the limit satisfies the magnetostatic condition $\boldsymbol{J} \times \boldsymbol{B} = \nabla p$. In Lions (1996), conditions upon the forcing to obtain this may be found. The general impression is that the convergence of the velocity to zero in the $L^2(\Omega)$-norm is the only solid result one can prove for this magnetic relaxation problem.

4. Conclusions

A classical model for magnetic relaxation involves the evolution of a viscous, non-resistive fluid under the MHD equations. It has always been assumed that this situation will lead to a static plasma where the magnetic field satisfies the magnetostatic equation, possibly with the presence of contact discontinuities such as current sheets that may prove useful in explaining magnetic reconnection phenomena. However, a proof of the existence of limits, even allowing the solutions to exist for all time, was lacking. It is shown here that even in the presence of a suitable forcing the velocity does tend to zero, so in a certain sense the limit is in fact static. The energy inequalities then show that the magnetic energy remains bounded, so the existence of weak sequential limits of the magnetic field is also assured. The existence of a unique limit, however, is not, except in the case when the velocity itself is absolutely integrable in time. There are good reasons for this: we show an example where there is no weak limit of the magnetic field as $t \to \infty$. It is true that this example involves the presence of a (decaying) forcing, but since this is arbitrarily small for large times, it seems possible that examples occur also in its absence. Thus the existence of magnetostatic states as limit of magnetic relaxation processes does not follow from first principles, and it must be proved in each case for every initial condition.

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